

# Weak $\omega$ -Categories as $\omega$ -Hypergraphs

Hiroyuki Miyoshi

Department of Computer Science, Kyoto Sangyo University

hxm@cc.kyoto-su.ac.jp

Toru Tsujishita

Department of Mathematics, Hokkaido University

tujisita@math.sci.hokudai.ac.jp

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## Abstract

In this paper, firstly, we introduce a higher-dimensional analogue of hypergraphs, namely  $\omega$ -hypergraphs. This notion is thoroughly flexible because unlike ordinary  $\omega$ -graphs, an  $n$ -dimensional edge called an  $n$ -cell has many sources and targets. Moreover, cells have polarity, with which pasting of cells is implicitly defined. As examples, we also give some known structures in terms of  $\omega$ -hypergraphs. Then we specify a special type of  $\omega$ -hypergraph, namely *directed  $\omega$ -hypergraphs*, which are made of cells with direction. Finally, based on them, we construct our *weak  $\omega$ -categories*. It is an  $\omega$ -dimensional variant of the weak  $n$ -categories given by Baez and Dolan [2]. We introduce  $\omega$ -identical,  $\omega$ -invertible and  $\omega$ -universal cells instead of universality and balancedness in [2]. The whole process of our definition is in parallel with the way of regarding categories as graphs with composition and identities.

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## 1 Introduction

J. Baez and J. Dolan recently proposed an important and impressive definition of weak  $n$ -categories[2]. They utilize nonstandard  $n$ -cells with not just one but many  $n - 1$ -cells as their domains for taming coherence conditions. Authors' primary motivation was to understand their idea along the famous slogan "categories are graphs with monoid structures". Thus they investigated a suitable notion of  $n$ - or  $\omega$ -dimensional graph-like structures which should include the underlying structures of Baez-Dolan-style weak  $\omega$ -categories.

In the way of pursuing such structures, they found a general notion of  $\omega$ -dimensional structures whose  $n$ -cells have many  $n - 1$ -cells not only in their domains but also in their codomains. This notion contains various categorical algebras:  $\omega$ -categories, bicategories, double categories, etc. Meanwhile, authors noticed that it can be thought of as a form of  $\omega$ -dimensional hypergraphs. Hypergraphs have been explored in mathematics[3], database theory[4], concurrency theory[6] and graph rewriting[13] as a device to represent complex notions. But their higher-dimensional extensions are still not known corresponding to  $n$ - or  $\omega$ -graphs for ordinary graphs. Therefore such structures are named  *$\omega$ -hypergraphs*<sup>1</sup>.

Thus the purpose of this paper is two-folded: One is to provide a general environment for representing various concepts, especially developing various category theories. Another is to give a definition of weak  $\omega$ -categories which respects saturatedness in the meaning of M. Makkai[9].

## 2 Trees and forests

Our main idea is to represent an  $n$ -cell as a tree with links and polarity. This is refinement of usual simplex (Figure 1). We start with the definition of trees and forests.

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<sup>1</sup>The definition of  $\omega$ -hypergraphs in this paper is not the most general form, because each node is shared by at most two hyperedges.

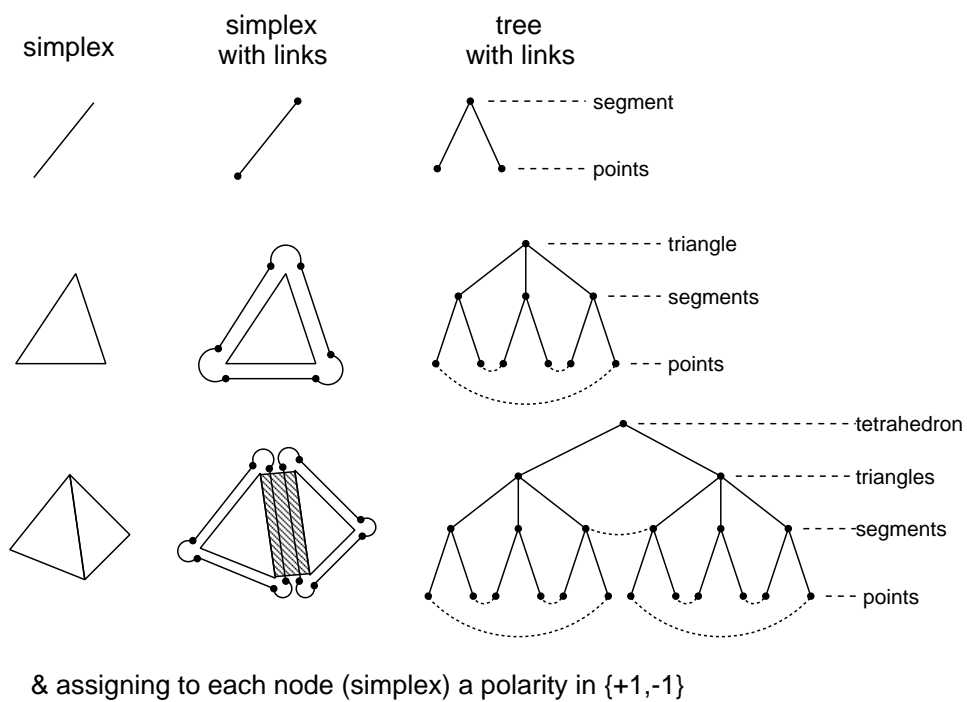


Figure 1: From simplices to trees with links and polarity

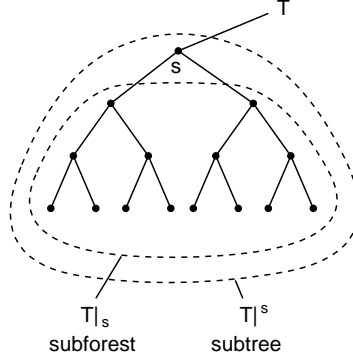


Figure 2: the subtree and subforest at  $s$

**Definition 2.1 ( $n$ -trees and  $n$ -forests)** For any natural number  $n \geq 0$ , an  $n$ -tree  $T$  is a triple  $\langle r^T, S^T, \pi^T \rangle$  consisting of

- $S_n^T = \{r^T\}$ , whose element  $r^T$  is called the *root* of  $T$ ;
- $S^T = \coprod_{0 \leq i \leq n} S_i^T$ , where  $S_i^T$  is a finite set whose elements are called  $i$ -nodes or simply *nodes*;
- $\pi^T = \prod_{0 \leq i \leq n-1} \pi_i^T$ , where  $\pi_i^T$  is a function from  $S_i^T$  to  $S_{i+1}^T$ .

Also, an  $n$ -forest  $F$  is a pair  $\langle S^F, \pi^F \rangle$ , consisting of

- $S^F = \prod_{0 \leq i \leq n} S_i^F$ , where  $S_i^F$  is a finite set of  $i$ -nodes;
- $\pi^F = \prod_{0 \leq i \leq n-1} \pi_i^F$ , where  $\pi_i^F$  is a function from  $S_i^F$  to  $S_{i+1}^F$ .

**Definition 2.2 (isomorphism of trees and forests)** For any natural number  $n \geq 0$ , a *homomorphism* of  $n$ -trees  $\sigma : T \rightarrow T'$  is a map from  $S^T$  to  $S^{T'}$  such that, for every  $x \in S_i^T$ ,  $\sigma(x) \in S_i^{T'}$  and  $\sigma \circ \pi^T = \pi^{T'} \circ \sigma$ . A homomorphism  $\sigma$  is an *isomorphism* when it is a bijection. A *homomorphism* and an *isomorphism* of  $n$ -forests are also defined in the same way.

**Definition 2.3 (subtrees and subforests)** For an  $n$ -tree  $T = \langle r^T, S^T, \pi^T \rangle$  and a  $k$ -node  $s$  ( $0 \leq k \leq n$ ), a *subtree* with the root  $s$  is defined as  $T|_s^s = \langle s, S^{T|_s^s}, \pi^{T|_s^s} \rangle$  where

- $S^{T|_s^s} = \prod_{0 \leq i \leq k} S_i^{T|_s^s}$ , where  $S_i^{T|_s^s} = \{t \in S_i^T \mid (\pi^T)^{k-i}(t) = s\}$ ;
- $\pi^{T|_s^s} = \prod_{0 \leq i \leq k-1} \pi_i^{T|_s^s}$  where  $\pi_i^{T|_s^s} = \pi_i|_{S_i^{T|_s^s}}$

And for a  $k$ -node  $s$  ( $1 \leq k \leq n$ ), a *subforest* under  $s$  is defined as  $T|_s = \langle S^{T|_s}, \pi^{T|_s} \rangle$  where

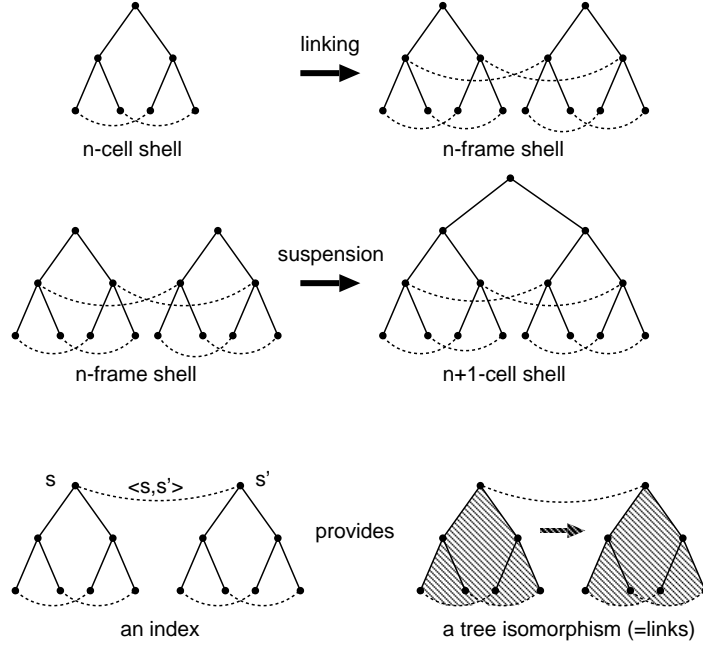


Figure 3: Mutually inductive construction of shells

- $S^{T|s} = \coprod_{0 \leq i \leq k-1} S_i^{T|s}$ , where  $S_i^{T|s} = \{t \in S_i^T \mid (\pi^T)^{k-i}(t) = s\}$ ;
- $\pi^{T|s} = \coprod_{0 \leq i \leq k-2} \pi_i^{T|s}$  where  $\pi_i^{T|s} = \pi_i|_{S_i^{T|s}}$

Also in the same way, for an  $n$ -forest  $F = \langle S^F, \pi^F \rangle$  and a  $k$ -node  $s$  ( $0 \leq k \leq n$ ), a *subtree* with the root  $s$  is defined as  $F|_s = \langle s, S^{F|s}, \pi^{F|s} \rangle$ , and for a  $k$ -node  $s$  ( $1 \leq k \leq n$ ), a *subforest* under  $s$  as  $F|_s = \langle S^{F|s}, \pi^{F|s} \rangle$ .

### 3 Shells

Shells play the same role as shape diagrams in the ordinary category theory. We will mutually inductively define a shell for each cell as a tree with polarity and links and one for each frame as a forest with polarity and links (Figure 3).

#### 3.1 the base case

**Definition 3.1 (0-cell shells and 0-frame shells)** A *0-cell shell*  $\theta$  is a singleton set with polarity. More precisely, it is  $\langle r^\theta, S^\theta, \emptyset, \epsilon^\theta, \emptyset, \emptyset \rangle$  where  $S^\theta = S_0^\theta = \{r^\theta\}$  and  $\epsilon^\theta$  is a function from  $S^\theta$  to  $\{-1, 1\}$ . Similarly, A *0-frame shell*  $\xi$  is a set with polarity, that is,  $\langle S^\xi, \emptyset, \epsilon^\xi, \emptyset, \emptyset \rangle$  where  $S^\xi = S_0^\xi$  and  $\epsilon^\xi$  is a function from  $S^\xi$  to  $\{-1, 1\}$ .

For the meaning of these definitions, see the following sections.

### 3.2 the induction step

Suppose that for the dimensions less than  $n$ , all stuff has already been defined.

**Definition 3.2** (*n-cell shells*) An *n-cell shell*  $\theta$  is

$$\langle r^\theta, S^\theta, \pi^\theta, \epsilon^\theta, \Upsilon^\theta, \{\sigma_{\langle s, s' \rangle}^\theta\}_{\langle s, s' \rangle \in \Upsilon^\theta} \rangle$$

consisting of the following data:

- $\underline{\theta} = \langle r^\theta, S^\theta, \pi^\theta \rangle$  is an  $n$ -tree, called the *base n-tree* of  $\theta$ ;
- *polarity*:  $\epsilon^\theta$  is a function from  $S^\theta$  to  $\{-1, 1\}$ ;
- *links*:  $\Upsilon^\theta = \coprod_{0 \leq i \leq n-2} \Upsilon_i^\theta$ , where  $\Upsilon_i^\theta \subset S_i^\theta \times S_i^\theta$ ;
- *linking isomorphisms*: for each  $\langle s, s' \rangle \in \Upsilon_i^\theta$ ,  $\sigma_{\langle s, s' \rangle}$  is a  $i$ -tree isomorphism from  $\underline{\theta}|^s$  to  $\underline{\theta}|^{s'}$ ,

which satisfy the following condition:

- *mutuality*:  $\theta|_{r^\theta} = \langle S^{\theta|_{r^\theta}}, \pi^{\theta|_{r^\theta}}, \epsilon^{\theta|_{r^\theta}}, \Upsilon^\theta, \{\sigma_{\langle s, s' \rangle}^\theta\}_{\langle s, s' \rangle \in \Upsilon^\theta} \rangle$  is an  $n-1$ -frame shell, where
  - $\langle S^{\theta|_{r^\theta}}, \pi^{\theta|_{r^\theta}} \rangle = \underline{\theta}|_{r^\theta}$
  - $\epsilon^{\theta|_{r^\theta}} = \epsilon^\theta|_{S^{\theta|_{r^\theta}}}$

**Definition 3.3** (*n-frame shells*) An *n-frame shell*  $\xi$  is

$$\langle S^\xi, \pi^\xi, \epsilon^\xi, \Upsilon^\xi, \{\sigma_{\langle s, s' \rangle}^\xi\}_{\langle s, s' \rangle \in \Upsilon^\xi} \rangle$$

consisting of the following data:

- $\underline{\xi} = \langle S^\xi, \pi^\xi \rangle$  is an  $n$ -forest, called the *base n-forest* of  $\xi$ ;
- *polarity*:  $\epsilon^\xi$  is a function from  $S^\xi$  to  $\{-1, 1\}$ ;
- *links*:  $\Upsilon^\xi = \coprod_{0 \leq i \leq n-1} \Upsilon_i^\xi$ , where  $\Upsilon_i^\xi \subset S_i^\xi \times S_i^\xi$ ;
- *linking isomorphisms*: for  $\langle s, s' \rangle \in \Upsilon_i^\xi$ ,  $\sigma_{\langle s, s' \rangle}$  is a  $i$ -tree isomorphism from  $\underline{\xi}|^s$  to  $\underline{\xi}|^{s'}$ ,

which satisfy the following conditions:

- *mutuality*: for any  $s \in S_n^\xi$ ,  $\xi|^s = \langle s, S^{\xi|^s}, \pi^{\xi|^s}, \epsilon^{\xi|^s}, \Upsilon^{\xi|^s}, \{\sigma_{\langle s, s' \rangle}^\xi\}_{\langle s, s' \rangle \in \Upsilon^{\xi|^s}} \rangle$  is an  $n$ -cell shell, where
  - $\langle s, S^{\xi|^s}, \pi^{\xi|^s} \rangle = \underline{\xi}|^s$

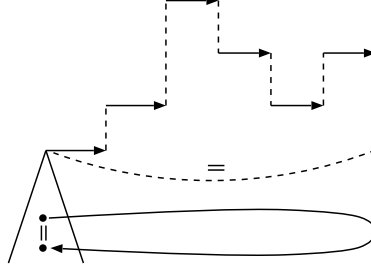


Figure 4: commutativity of links

- $\epsilon^{\xi|s} = \epsilon^{\xi}|_{S^{\xi|s}}$ ;
- $\Upsilon^{\xi|s} = \coprod_{0 \leq i \leq n-2} \Upsilon_i^{\xi|s}$ , where  $\Upsilon_i^{\xi|s} = \Upsilon_i^{\xi}|_{S_i^{\xi|s} \times S_i^{\xi|s}}$
- *bijectivity*: if  $\langle s, t \rangle, \langle s, t' \rangle \in \Upsilon_{n-1}^{\xi}$ , then  $t = t'$ ; and if  $\langle s, t \rangle, \langle s', t \rangle \in \Upsilon_{n-1}^{\xi}$ , then  $s = s'$ ;
- *involution*: if  $\langle s, t \rangle \in \Upsilon_{n-1}^{\xi}$ , then  $\langle t, s \rangle \in \Upsilon_{n-1}^{\xi}$  and  $\sigma_{\langle s, t \rangle}^{\xi} \circ \sigma_{\langle t, s \rangle}^{\xi} = \text{Id}$  and  $\sigma_{\langle t, s \rangle}^{\xi} \circ \sigma_{\langle s, t \rangle}^{\xi} = \text{Id}$ ;
- *conjugation*: if  $\langle s, s' \rangle \in \Upsilon_{n-1}^{\xi}$  and  $t \in S^{\xi}$  such that  $(\pi^{\xi})^i(t) = s$  for some  $i \geq 0$ , then  $\epsilon(t)\epsilon(\sigma_{\langle s, s' \rangle}^{\xi}(t)) = -1$  (this implies *anti-reflexivity*: if  $\langle s, s' \rangle \in \Upsilon_{n-1}^{\xi}$ , then  $s \neq s'$ );
- *correspondence of links*: if  $\langle s, s' \rangle \in \Upsilon_{n-1}^{\xi}$  and  $\langle t, t' \rangle \in \Upsilon_i^{\xi}$  for some  $i \leq n-3$  and  $(\pi^{\xi})^{n-i}(t) = (\pi^{\xi})^{n-i}(t') = s$ , then  $\langle \sigma_{\langle s, s' \rangle}^{\xi}(t), \sigma_{\langle s, s' \rangle}^{\xi}(t') \rangle \in \Upsilon_i^{\xi}$  and  $\sigma_{\langle t, t' \rangle}^{\xi} \circ \sigma_{\langle s, s' \rangle}^{\xi} = \sigma_{\langle \sigma_{\langle s, s' \rangle}^{\xi}(t), \sigma_{\langle s, s' \rangle}^{\xi}(t') \rangle}^{\xi} \circ \sigma_{\langle t, t' \rangle}^{\xi}$ ;
- *commutativity of links*: for  $k \geq 2$  and  $\langle s'_1, s_2 \rangle, \langle s'_2, s_3 \rangle, \dots, \langle s'_{k-1}, s_k \rangle, \langle s'_k, s_1 \rangle$  in  $\Upsilon^{\xi}$  such that  $\pi^{\xi}(s_i) = s'_i$  or  $\pi^{\xi}(s'_i) = s_i$ , if  $s' = (\sigma_{\langle s'_k, s_1 \rangle}^{\xi} \circ \sigma_{\langle s'_{k-1}, s_k \rangle}^{\xi} \circ \dots \circ \sigma_{\langle s'_2, s_3 \rangle}^{\xi} \circ \sigma_{\langle s'_1, s_2 \rangle}^{\xi})(s)$  is defined, then  $s' = s$ ; (that is, if  $s'_i$  is of the smallest level between  $s'_1, \dots, s'_k$ , then the composition of such isomorphisms as above  $\sigma_{\langle s'_q, s_i \rangle}^{\xi} \circ \dots \circ \sigma_{\langle s'_i, s_p \rangle}^{\xi}$  is defined and is the identity homomorphism of the subtree at  $s'_i$ ).
- *closedness*: for every  $s \in S_{n-1}^{\xi}$ , there exists a(n unique) node  $s' \in S_{n-1}^{\xi}$  such that  $\langle s, s' \rangle \in \Upsilon_{n-1}^{\xi}$ .

Closedness means *globularity* of higher dimensional cells. Note that every  $t \in S^{\xi}$  is in  $S^{\xi|s}$  for just one  $s \in S_n^{\xi}$ ; and also every  $\langle t, t' \rangle \in \Upsilon_i^{\xi}$  for  $i \leq n-2$  is in  $\Upsilon_i^{\xi|s}$  for just one  $s \in S_n^{\xi}$ . The latter is due to the closedness of frame shells at lower levels.

**Proposition 3.1** For a cell shell  $\theta$ , if  $\langle s, s' \rangle \in \Upsilon^\theta$ , then  $\pi^i(s) = \pi^i(s')$  where  $i = 1$  or  $2$ .

**Remark 3.1** Thus the situation of the correspondence of links for  $k = 2$  occur only when  $l_1 = l_2 = 1$  and either the parents of  $s'_1$  and  $s_2$  or those of  $s_2$  and  $s'_1$  are the same. And for an  $n$ -cell shell  $\theta$  in  $n$ -frame shell, an *outer link*  $\langle s, s' \rangle$  of which  $s$  or  $s'$  is not in  $S_\theta$ , must be an  $n - 1$ -link.

**Definition 3.4** ( $\cong_n, (-)^*$ ) For two  $n$ -frame shells

$$\begin{aligned}\xi &= \langle S^\xi, \pi^\xi, \epsilon^\xi, \Upsilon^\xi, \{\sigma_{\langle s, s' \rangle}^\xi\}_{\langle s, s' \rangle \in \Upsilon^\xi} \rangle \quad \text{and} \\ \xi' &= \langle S^{\xi'}, \pi^{\xi'}, \epsilon^{\xi'}, \Upsilon^{\xi'}, \{\sigma_{\langle s, s' \rangle}^{\xi'}\}_{\langle s, s' \rangle \in \Upsilon^{\xi'}} \rangle,\end{aligned}$$

an *isomorphism*  $f$  from  $\xi$  to  $\xi'$  is an  $n$ -forest isomorphism (with its inverse  $f^{-1}$ ) such that

- $\epsilon^\xi(s) = \epsilon^{\xi'}(f(s)) \quad (\Leftrightarrow \epsilon^{\xi'}(t) = \epsilon^\xi(f^{-1}(t)))$ ;
- if  $\langle s, s' \rangle \in \Upsilon^\xi$ , then  $\langle f(s), f(s') \rangle \in \Upsilon^{\xi'} \quad (\Leftrightarrow \text{if } \langle t, t' \rangle \in \Upsilon^{\xi'}, \text{ then } \langle f^{-1}(t), f^{-1}(t') \rangle \in \Upsilon^\xi)$ ;
- $f \circ \sigma_{\langle s, s' \rangle}^\xi = \sigma_{\langle f(s), f(s') \rangle}^{\xi'} \circ f \quad (\Leftrightarrow f^{-1} \circ \sigma_{\langle t, t' \rangle}^{\xi'} = \sigma_{\langle f^{-1}(t), f^{-1}(t') \rangle}^\xi \circ f^{-1})$ .

When an isomorphism  $f$  from  $\xi$  to  $\xi'$  exists, we say that  $\xi$  is *isomorphic* to  $\xi'$ , and write  $f : \xi \cong_n \xi'$ ,  $\xi \cong_n \xi'$ , or simply  $\xi \cong \xi'$ . Obviously  $\cong_n$  is an equivalence relation.

For an  $n$ -frame shell  $\xi$ ,  $(\xi)^*$  is defined as  $\langle S^\xi, \pi^\xi, \epsilon^{(\xi)*}, \Upsilon^\xi, \{\sigma_{\langle s, s' \rangle}^\xi\}_{\langle s, s' \rangle \in \Upsilon^\xi} \rangle$  where  $\epsilon^{(\xi)*}(s) = -\epsilon^\xi(s)$ . It is easy to check well-definedness, that is,  $(\xi)^*$  is in fact an  $n$ -frame shell, and  $((\xi)^*)^* = \xi$ .

An  $n$ -cell shell can be seen as an  $n$ -frame shell. Thus we can define isomorphisms between  $n$ -cell shells similarly.

## 4 Diagrams and $\omega$ -hypergraphs

Cell diagrams and frames are mutually inductively defined.

**Definition 4.1** (*i*-cell) We prepare a set of  $i$ -cells for each  $i \in \mathbb{N} \cup \{0\}$ :

- $\Sigma_i = \Sigma_{i,-1} \amalg \Sigma_{i,1}$ ,
- a bijection  $(-)^* : \Sigma_i \rightarrow \Sigma_i$  such that for  $c \in \Sigma_{i,k}$  with  $k \in \{1, -1\}$ ,  $c^* \in \Sigma_{i,-k}$  and  $(c^*)^* = c$ .

Elements of  $\Sigma_i$  are called *i*-cells; those of  $\Sigma_{i,1}$  *positive i*-cells; and those of  $\Sigma_{i,-1}$  *negative i*-cells.  $c^*$  is called the *conjugate* of  $c$ .



## 4.1 the base case

**Definition 4.2 (0-hypergraph, 0-cell diagram and 0-frame)** For consistency, let  $\mathbf{Frm}_{-1}$  be  $\{\emptyset\}$ , the only one  $-1$ -frame isomorphism the empty function  $\emptyset : \emptyset \rightarrow \emptyset$  and  $\partial_0 : \Sigma_0 \rightarrow \mathbf{Frm}_{-1}$  the unique function. A *0-hypergraph* is  $\langle \Sigma_0, \partial_0 \rangle$ . A *0-cell diagram*  $\eta$  is  $\langle r^\eta, S^\eta, \emptyset, \epsilon^\eta, \emptyset, \emptyset, \lambda^\eta, \{\rho_{r^\eta}^\eta\}_{r^\eta \in S^\eta} \rangle$ , where  $\underline{\eta} = \langle r^\eta, S^\eta, \emptyset, \epsilon^\eta, \emptyset, \emptyset \rangle$  is a 0-cell shell,  $\lambda^\eta$  is a function from  $S^\eta$  to  $\Sigma_0$  such that  $\lambda^\eta(r^\eta) \in \Sigma_{0, \epsilon^\eta(r^\eta)}$ , and  $\rho_{r^\eta}^\eta$  is the empty function. A *0-frame diagram*, or simply a *0-frame*,  $\zeta$  is  $\langle S^\zeta, \emptyset, \epsilon^\zeta, \emptyset, \emptyset, \lambda^\zeta, \{\rho_s^\zeta\}_{s \in S^\zeta} \rangle$ , where  $\underline{\zeta} = \langle S^\zeta, \emptyset, \epsilon^\zeta, \emptyset, \emptyset \rangle$  is a 0-frame shell,  $\lambda^\zeta$  is a function from  $S^\zeta$  to  $\Sigma_0$  such that for any  $s \in S^\zeta$ ,  $\lambda^\zeta(s) \in \Sigma_{0, \epsilon^\zeta(s)}$ , and each  $\rho_s^\zeta$  is the empty function. A *0-frame isomorphism* from  $\zeta$  to  $\zeta'$  is a 0-frame shell isomorphism  $f : \underline{\zeta} \rightarrow \underline{\zeta'}$  (in fact, a bijection from  $S^\zeta$  to  $S^{\zeta'}$ ) satisfying  $\lambda^\zeta = \lambda^{\zeta'} \circ f$ .  $\mathbf{Frm}_0$  is the set of 0-frames.

## 4.2 the induction step

Suppose that  $n \geq 1$  and that for the dimensions less than  $n$ , all stuff has already been defined.

**Definition 4.3 (boundary of  $n$ -cells)** As a parameter of definitions, a function  $\partial_n : \Sigma_n \rightarrow \mathbf{Frm}_{n-1}$  satisfying  $(\partial_n(c))^* = \partial_n(c^*)$  are given (for the  $n-1$  dimension,  $(-)^*$  for frames have been defined).  $\partial_n(c)$  is called the *boundary* of  $c$ .

**Definition 4.4 ( $n$ -hypergraph)** An  *$n$ -hypergraph*  $G = \langle \Sigma, \partial \rangle$  consists of

- $\Sigma = \coprod_{0 \leq i \leq n} \Sigma_i$ , and
- $\partial = \coprod_{1 \leq i \leq n} \partial_i$ .

**Definition 4.5 ( $n$ -cell diagram)** An  *$n$ -cell diagram*  $\eta$  is

$$\langle r^\eta, S^\eta, \pi^\eta, \epsilon^\eta, \Upsilon^\eta, \{\sigma_{\langle s, s' \rangle}^\eta\}_{\langle s, s' \rangle \in \Upsilon^\eta}, \lambda^\eta, \{\rho_s^\eta\}_{s \in S^\eta} \rangle$$

where

- $\underline{\eta} = \langle r^\eta, S^\eta, \pi^\eta, \epsilon^\eta, \Upsilon^\eta, \{\sigma_{\langle s, s' \rangle}^\eta\}_{\langle s, s' \rangle \in \Upsilon^\eta} \rangle$  is an  $n$ -cell shell, called the *base  $n$ -cell shell* of  $\eta$ ;
- *assignment of cells*:  $\lambda^\eta = \coprod_{0 \leq i \leq n} \lambda_i^\eta$ , where  $\lambda_i^\eta$  is a function from  $S_i^\eta$  to  $\Sigma_i$  such that for any  $s \in S_i$ ,  $\lambda_i^\eta(s) \in \Sigma_{i, \epsilon(s)}$
- *identification in boundaries*: for  $s \in S_i^\eta$ ,  $\rho_s^\eta$  is an  $i-1$ -frame isomorphism from  $\eta|_s$  to  $\partial_i(\lambda_i^\eta(s))$

which satisfy the following conditions:

- *mutuality*:  $\eta|_{r^\eta} = \langle S^{\eta|_{r^\eta}}, \pi^{\eta|_{r^\eta}}, \epsilon^{\eta|_{r^\eta}}, \Upsilon^\eta, \{\sigma_{\langle s, s' \rangle}^\eta\}_{\langle s, s' \rangle \in \Upsilon^\eta}, \lambda^{\eta|_{r^\eta}}, \{\rho_s^\eta\}_{s \in S^{\eta|_{r^\eta}}} \rangle$  is an  $n-1$ -frame, where

- $\langle S^{\eta|_{r^{\eta}}}, \pi^{\eta|_{r^{\eta}}}, \epsilon^{\eta|_{r^{\eta}}}, \Upsilon^{\eta}, \{\sigma_{\langle s, s' \rangle}^{\eta}\}_{\langle s, s' \rangle \in \Upsilon^{\eta}} \rangle = \underline{\eta}|_{r^{\eta}}$
- $\lambda^{\eta|_{r^{\eta}}} = \lambda^{\eta}|_{S^{\eta|_{r^{\eta}}}}$ .

**Definition 4.6** (*n-frame*) *n-frame diagram* or *n-frame*  $\zeta$  is

$$\langle S^{\zeta}, \pi^{\zeta}, \epsilon^{\zeta}, \Upsilon^{\zeta}, \{\sigma_{\langle s, s' \rangle}^{\zeta}\}_{\langle s, s' \rangle \in \Upsilon^{\zeta}}, \lambda^{\zeta}, \{\rho_s^{\zeta}\}_{s \in S^{\zeta}} \rangle$$

where

- $\underline{\zeta} = \langle S^{\zeta}, \pi^{\zeta}, \epsilon^{\zeta}, \Upsilon^{\zeta}, \{\sigma_{\langle s, s' \rangle}^{\zeta}\}_{\langle s, s' \rangle \in \Upsilon^{\zeta}} \rangle$  is an *n-frame shell*, called the *base n-frame shell* of  $\zeta$ ;
- *assignment of cells*:  $\lambda^{\zeta} = \coprod_{0 \leq i \leq n} \lambda_i^{\zeta}$ , where  $\lambda_i^{\zeta}$  is a function from  $S_i^{\zeta}$  to  $\Sigma_i$  such that for any  $s \in S_i$ ,  $\lambda_i(s) \in \Sigma_{i, \epsilon(s)}$
- *identification in boundaries*: for  $s \in S_i^{\zeta}$ ,  $\rho_s^{\zeta}$  is an  $i - 1$ -frame isomorphism from  $\zeta|_s$  to  $\partial_i(\lambda_i^{\zeta}(s))$

which satisfy the following conditions:

- *mutuality*: for every  $s \in S_n^{\zeta}$ ,  $\zeta|_s = \langle s, S^{\zeta|_s}, \pi^{\zeta|_s}, \epsilon^{\zeta|_s}, \Upsilon^{\zeta|_s}, \{\sigma_{\langle s, s' \rangle}^{\zeta}\}_{\langle s, s' \rangle \in \Upsilon^{\zeta|_s}}, \lambda^{\zeta|_s}, \{\rho_t^{\zeta}\}_{t \in S^{\zeta|_s}} \rangle$  is an *n-cell diagram*, where
  - $\langle s, S^{\zeta|_s}, \pi^{\zeta|_s}, \epsilon^{\zeta|_s}, \Upsilon^{\zeta|_s}, \{\sigma_{\langle s, s' \rangle}^{\zeta}\}_{\langle s, s' \rangle \in \Upsilon^{\zeta|_s}} \rangle = \underline{\zeta}|_s$
  - $\lambda^{\zeta|_s} = \lambda^{\zeta}|_{S^{\zeta|_s}}$
- *compatibility on links*:
  - for  $\langle s, s' \rangle \in \Upsilon_{n-1}^{\zeta}$ ,  $\lambda(s) = (\lambda(s'))^*$  and
  - for  $\langle s, s' \rangle \in \Upsilon_{n-1}^{\zeta}$  and  $t \in S^{\zeta|_s}$ ,  $\rho_s^{\zeta}(t) = \rho_{s'}^{\zeta}(\sigma_{\langle s, s' \rangle}(t))$

**Proposition 4.1** For  $\langle s, s' \rangle \in \Upsilon_{n-1}^{\zeta}$  and  $t \in S_k^{\zeta}$  for some  $k \leq n - 1$  such that  $(\pi^{\zeta})^{n-k-1}(t) = s$ ,  $(\lambda(t))^* = \lambda(\sigma_{\langle s, s' \rangle}^{\zeta}(t))$ .

**Proof** It is induced from the compatibility on links and the definition of  $(-)^*$  for  $n - 2$ -frames.  $\square$

**Definition 4.7** ( $\cong_n, \mathbf{Frm}_n, (-)^*$ ) For two *n-frames*

$$\begin{aligned} \zeta &= \langle S^{\zeta}, \pi^{\zeta}, \epsilon^{\zeta}, \Upsilon^{\zeta}, \{\sigma_{\langle s, s' \rangle}^{\zeta}\}_{\langle s, s' \rangle \in \Upsilon^{\zeta}}, \lambda^{\zeta}, \{\rho_s^{\zeta}\}_{s \in S^{\zeta}} \rangle \quad \text{and} \\ \zeta' &= \langle S^{\zeta'}, \pi^{\zeta'}, \epsilon^{\zeta'}, \Upsilon^{\zeta'}, \{\sigma_{\langle s, s' \rangle}^{\zeta'}\}_{\langle s, s' \rangle \in \Upsilon^{\zeta'}}, \lambda^{\zeta'}, \{\rho_s^{\zeta'}\}_{s \in S^{\zeta'}} \rangle, \end{aligned}$$

an *isomorphism*  $f$  from  $\zeta$  to  $\zeta'$  is an isomorphism of *n-frame shells*  $f : \underline{\zeta} \rightarrow \underline{\zeta}'$  (with its inverse  $f^{-1}$ ) such that

- for  $s \in S^{\zeta}$ ,  $\lambda^{\zeta}(s) = \lambda^{\zeta'}(f(s)) \quad (\Leftrightarrow \lambda^{\zeta'}(s') = \lambda^{\zeta}(f^{-1}(s')))$ ,

- for  $s \in S^\zeta$  and  $t \in S^{\zeta|_s}$ ,  $\rho_s^\zeta(t) = \rho_{f(s)}^{\zeta'}(f(t)) \quad (\Leftrightarrow \rho_{s'}^{\zeta'}(t') = \rho_{f^{-1}(s')}(f^{-1}(t')))$ .

When an isomorphism  $f$  from  $\zeta$  to  $\zeta'$  exists, we say that  $\zeta$  is *isomorphic* to  $\zeta'$ , and write  $f : \zeta \cong_n \zeta'$ ,  $\zeta \cong_n \zeta'$ , or simply  $\zeta \cong \zeta'$ . Obviously  $\cong_n$  is an equivalence relation. The collection of all  $n$ -frames is denoted by  $\mathbf{Frm}_n$ . For an  $n$ -frame  $\zeta$ ,  $(\zeta)^*$  is defined as  $\langle S^\zeta, \pi^\zeta, \epsilon^{(\zeta)*}, \Upsilon^\zeta, \{\sigma_{\langle s, s' \rangle}^\zeta\}_{\langle s, s' \rangle \in \Upsilon^\zeta}, \lambda^{(\zeta)*} \rangle$  where  $\epsilon^{(\zeta)*}(s) = -\epsilon^\zeta(s)$  and  $\lambda^{(\zeta)*}(s) = (\lambda^\zeta(s))^*$ . It is easy to check well-definedness, that is,  $(\zeta)^*$  is in fact an  $n$ -frame, and  $((\zeta)^*)^* = \zeta$ .

An  $n$ -cell diagram can be seen as an  $n$ -frame. Thus we can define an *isomorphism* between  $n$ -cell diagrams similarly.

**Remark 4.1** Indeed, conditions for  $\rho$  in the definitions of cell diagrams and frames *ensure* the commutativity of links and other commutativity of their base shells (it is easy to check this). Therefore if we use shells only for diagrams, we need not introduce such commutativity. A main purpose to do it is to treat closure operations for shells. Due to commutativity, a closure becomes unique in a sense.

### 4.3 $\omega$ -hypergraphs

**Definition 4.8 ( $\omega$ -hypergraph)** An  $\omega$ -hypergraph  $G = \langle \Sigma, \partial \rangle$  consists of

- $\Sigma = \coprod_{0 \leq i} \Sigma_i$ , and
- $\partial = \coprod_{1 \leq i} \partial_i$ .

**Remark 4.2** Boundaries  $\partial_i$  depend on frames in the previous step of the inductive definition. Therefore as pointed out in [11], to formalize the definition of  $\omega$ -hypergraphs in a logical system, we need a sort of *dependent choice* axiom,  $GDC_\tau$  in [5] §4.4.3 or  $DC_1$  in [1] §8.2.3. The strength of this is in between the countable axiom of choice and the full axiom of choice .

## 5 Pasting diagrams and their closures

**Definition 5.1 ( $n$ -pasting shells)** An  $n$ -pasting shell consists of the same data and conditions as an  $n$ -frame shell, but at the last induction step, the closedness condition is not required. That is,  $n - 1$ -nodes which do not appear in  $\Upsilon_{n-1}^\zeta$  are allowed. We call them *open* nodes of the pasting shell. An  $n$ -pasting shell is *positive* or *negative* if for all  $s \in S_n^\zeta$ ,  $\epsilon_n^\zeta(s) = 1$  or  $-1$ , respectively.

**Definition 5.2 ( $n$ -pasting diagrams)** An  $n$ -pasting diagram  $\zeta$  is defined in the same way as  $n$ -frame, but  $\zeta$  is an  $n$ -pasting shell instead of an  $n$ -frame shell.  $\cong_n$ ,  $\mathbf{PD}_n$ ,  $(-)^*$  is also defined similarly. An  $n$ -pasting diagram is *positive* or *negative* if the base  $n$ -pasting shell is positive or negative, respectively.

**Lemma 5.1** Consider an  $n$ -pasting shell  $\xi = \langle S^\xi, \pi^\xi, \epsilon^\xi, \Upsilon^\xi, \{\sigma_{\langle s, s' \rangle}^\xi\}_{\langle s, s' \rangle \in \Upsilon^\xi} \rangle$ . Let a condition  $\Psi(y_0, y_1, \dots, y_m; x_0, x_1, \dots, x_m)$  ( $1 \leq m$ ) be abbreviated that

- $\langle x_0, x_1 \rangle, \langle x_2, x_3 \rangle, \dots, \langle x_{m-1}, x_m \rangle \in \Upsilon_{n-2}^\xi$ ,
- $\langle y_0, y_1 \rangle, \langle y_2, y_3 \rangle, \dots, \langle y_{m-1}, y_m \rangle \in \Upsilon_{n-1}^\xi$ ,
- $\pi^\xi(x_i) = y_i$  and
- $\sigma_{\langle x_{m-1}, x_m \rangle}^\xi \circ \dots \circ \sigma_{\langle x_2, x_3 \rangle}^\xi \circ \sigma_{\langle y_1, y_2 \rangle}^\xi \circ \sigma_{\langle x_0, x_1 \rangle}^\xi(x_0) = x_m$ .

Note that same nodes may be duplicated in parameters of  $\Psi$ ; in particular,  $y_0$  may be equal to  $y_m$ . Then

1. For every  $y_0, y_1, \dots, y_m$  and  $x_0, x_1, \dots, x_m$  satisfying  $\Psi(y_0, y_1, \dots, y_m; x_0, x_1, \dots, x_m)$ , we have  $\Psi(y_m, y_{m-1}, \dots, y_0; x_m, x_{m-1}, \dots, x_0)$ .
2. For every open node  $y_0$  and its child  $x_0$ , there uniquely exist  $y_0, y_1, \dots, y_m$  and  $x_0, x_1, \dots, x_m$  satisfying  $\Psi(y_0, y_1, \dots, y_m; x_0, x_1, \dots, x_m)$  and that  $y_m$  is an open node ( $y_1, \dots, y_{m-1}$  are not open by the second condition of  $\Psi$ ).

**Proof** (1) Trivial from the conditions of frame shells. (2) Starting from  $y_0$  and  $x_0$ , we can uniquely fix a required sequence  $y_0, x_0, x_1, y_1, y_2, x_2, \dots$  by the following process: For  $x_{2i}$  ( $0 \leq i$ ),  $x_{2i+1}$  is uniquely determined by the bijectivity of links; then  $\pi^\xi(x_{2i+1}) = y_{2i+1}$  and for  $y_{2i+1}$ ,  $y_{2i+2}$  is again uniquely determined by the bijectivity of links; therefore for  $x_{2i+1}$ ,  $\sigma_{\langle y_{2i+1}, y_{2i+2} \rangle}(x_{2i+1}) = x_{2i+2}$  is also unique. Next, we show that this process necessarily gets to an open node. Every link  $\langle x, x' \rangle \in \Upsilon_{n-2}^\xi$  appears at most once in the process because for a link to appear twice means the existence of a link  $\langle y, y_0 \rangle \in \Upsilon_{n-1}^\xi$  for some  $y$ , and this contradicts that  $y_0$  is open. Since  $S_{n-2}$  is finite and so is  $\Upsilon_{n-2}^\xi$ , the process starting from an open node  $y_0$  reaches an open node  $y_m$  for a finite  $m$  and stops there.  $\square$

**Proposition 5.2 (closer and closure of an  $n$ -pasting shell)** For any  $n$ -pasting shell  $\xi = \langle S^\xi, \pi^\xi, \epsilon^\xi, \Upsilon^\xi, \{\sigma_{\langle s, s' \rangle}^\xi\}_{\langle s, s' \rangle \in \Upsilon^\xi} \rangle$ , we can construct a closer of  $\xi$ , an  $n$ -cell shell  $\hat{\xi} = \langle r^{\hat{\xi}}, S^{\hat{\xi}}, \pi^{\hat{\xi}}, \epsilon^{\hat{\xi}}, \Upsilon^{\hat{\xi}}, \{\sigma_{\langle s, s' \rangle}^{\hat{\xi}}\}_{\langle s, s' \rangle \in \Upsilon^{\hat{\xi}}} \rangle$ , and a closure of  $\xi$ , an  $n$ -frame shell  $\bar{\xi} = \langle S^{\bar{\xi}}, \pi^{\bar{\xi}}, \epsilon^{\bar{\xi}}, \Upsilon^{\bar{\xi}}, \{\sigma_{\langle s, s' \rangle}^{\bar{\xi}}\}_{\langle s, s' \rangle \in \Upsilon^{\bar{\xi}}} \rangle$  uniquely up to isomorphisms and polarity as follows: Let  $\{s_0, s_1, \dots, s_k\}$  be the set of open nodes of  $\xi$ . For each  $s_l$ , we prepare an  $n-1$ -cell shell  $\tau_l = \langle t_l, S^{\tau_l}, \pi^{\tau_l}, \epsilon^{\tau_l}, \Upsilon^{\tau_l}, \{\sigma_{\langle t, t' \rangle}^{\tau_l}\}_{\langle t, t' \rangle \in \Upsilon^{\tau_l}} \rangle$ , isomorphic to  $(\xi|^{s_l})^*$  via an isomorphism  $f_l : \tau_l \rightarrow (\xi|^{s_l})^*$ . Then the components of the closer  $\hat{\xi}$  are:

- $S_n^{\hat{\xi}} = \{r^{\hat{\xi}}\}$  where  $\{r^{\hat{\xi}}\}$  is a singleton set, and  $S_i^{\hat{\xi}} = \coprod_{0 \leq l \leq k} S_i^{\tau_l}$  for  $0 \leq i \leq n-1$ ,

- $\pi_{n-1}^{\hat{\xi}} = \pi_{r^{\hat{\xi}}}$  where  $\pi_{r^{\hat{\xi}}}(t_l) = r^{\hat{\xi}}$  for  $0 \leq l \leq k$ , and  $\pi_i^{\hat{\xi}} = \prod_{0 \leq l \leq k} \pi_i^{\tau_l}$  for  $0 \leq i \leq n-2$ ,
- $\epsilon^{\hat{\xi}} = (\prod_{0 \leq l \leq k} \epsilon^{\tau_l}) \amalg \epsilon_{r^{\hat{\xi}}}$  where  $\epsilon_{r^{\hat{\xi}}}(r^{\hat{\xi}}) = -1$  (the negative closer) or 1 (the positive closer),
- $\Upsilon^{\hat{\xi}} = (\prod_{0 \leq l \leq k} \Upsilon^{\tau_l}) \amalg \{ \langle f_l^{-1}(x), f_{l'}^{-1}(x') \rangle \mid \Phi(s_l, s_{l'}; x, x') \}$ , where  $\Phi(s_l, s_{l'}; x, x')$  is abbreviated that there exist  $y_0 = s_l, y_1, y_2, \dots, y_m = s_{l'}$  and  $x_0 = x, x_1, x_2, \dots, x_m = x'$  satisfying  $\Psi(y_0, y_1, \dots, y_m; x_0, x_1, \dots, x_m)$ ,
- $\{\sigma_{\langle s, s' \rangle}^{\hat{\xi}}\}_{\langle s, s' \rangle \in \Upsilon^{\hat{\xi}}}$  is defined as
  - $\sigma_{\langle t, t' \rangle}^{\hat{\xi}} = \sigma_{\langle t, t' \rangle}^{\tau_l}$  for  $\langle t, t' \rangle \in \Upsilon^{\tau_l}$ ,
  - $\sigma_{\langle f_l^{-1}(x), f_{l'}^{-1}(x') \rangle}^{\hat{\xi}} = f_{l'}^{-1} \circ \sigma_{\langle x_m, x \rangle}^{\xi} \circ \dots \circ \sigma_{\langle x_2, x_3 \rangle}^{\xi} \circ \sigma_{\langle y_1, y_2 \rangle}^{\xi} \circ \sigma_{\langle x_0, x_1 \rangle}^{\xi} \circ f_l$ .

and the components of the closure  $\bar{\xi}$  are:

- $S^{\bar{\xi}} = S^{\xi} \amalg S^{\hat{\xi}}$ ,
- $\pi^{\bar{\xi}} = \pi^{\xi} \amalg \pi^{\hat{\xi}}$ ,
- $\epsilon^{\bar{\xi}} = \epsilon^{\xi} \amalg \epsilon^{\hat{\xi}}$ ,
- $\Upsilon^{\bar{\xi}} = \Upsilon^{\xi} \amalg \{ \langle s_l, t_l \rangle, \langle t_l, s_l \rangle \mid 0 \leq l \leq k \} \amalg \Upsilon^{\hat{\xi}}$ ,
- $\{\sigma_{\langle s, s' \rangle}^{\bar{\xi}}\}_{\langle s, s' \rangle \in \Upsilon^{\bar{\xi}}}$  is defined as
  - $\sigma_{\langle s, s' \rangle}^{\bar{\xi}} = \sigma_{\langle s, s' \rangle}^{\xi}$  for  $\langle s, s' \rangle \in \Upsilon^{\xi}$ ,
  - $\sigma_{\langle t_l, s_l \rangle}^{\bar{\xi}} = f_l$  and  $\sigma_{\langle s_l, t_l \rangle}^{\bar{\xi}} = f_l^{-1}$ ,
  - $\sigma_{\langle t, t' \rangle}^{\bar{\xi}} = \sigma_{\langle t, t' \rangle}^{\hat{\xi}}$  for  $\langle t, t' \rangle \in \Upsilon^{\hat{\xi}}$ .

**Proof** First, we will check conditions for  $\langle f_l^{-1}(x), f_{l'}^{-1}(x') \rangle$  and  $\sigma_{\langle f_l^{-1}(x), f_{l'}^{-1}(x') \rangle}^{\bar{\xi}}$ . Other parts are rather easy:

- The mutuality condition is obvious. The bijectivity condition is shown by the Lemma 5.1 (2) and the involution condition by the Lemma 5.1 (1).
- The conjugation condition is derived from the following results:  $\epsilon^{\xi}(x_{2i})\epsilon^{\xi}(x_{2i+1}) = -1$  by the conjugation of  $\xi$ ,  $\epsilon^{\xi}(x_{2i+1})\epsilon^{\xi}(x_{2i+2}) = -1$  by the definition of  $\sigma_{\langle y_{2i+1}, y_{2i+2} \rangle}^{\xi}$ ,  $\epsilon^{\tau_l}(f_l^{-1}(x))\epsilon^{\xi}(x) = -1$  and  $\epsilon^{\tau_{l'}}(f_{l'}^{-1}(x'))\epsilon^{\xi}(x') = -1$ .
- The correspondence of links condition for  $\langle f_l^{-1}(x), f_{l'}^{-1}(x') \rangle$  is shown by chaining the correspondence of links in  $\xi$  and commutativity of  $f, f'$ .
- The closedness condition is straightforward from the Lemma 5.1 (2).

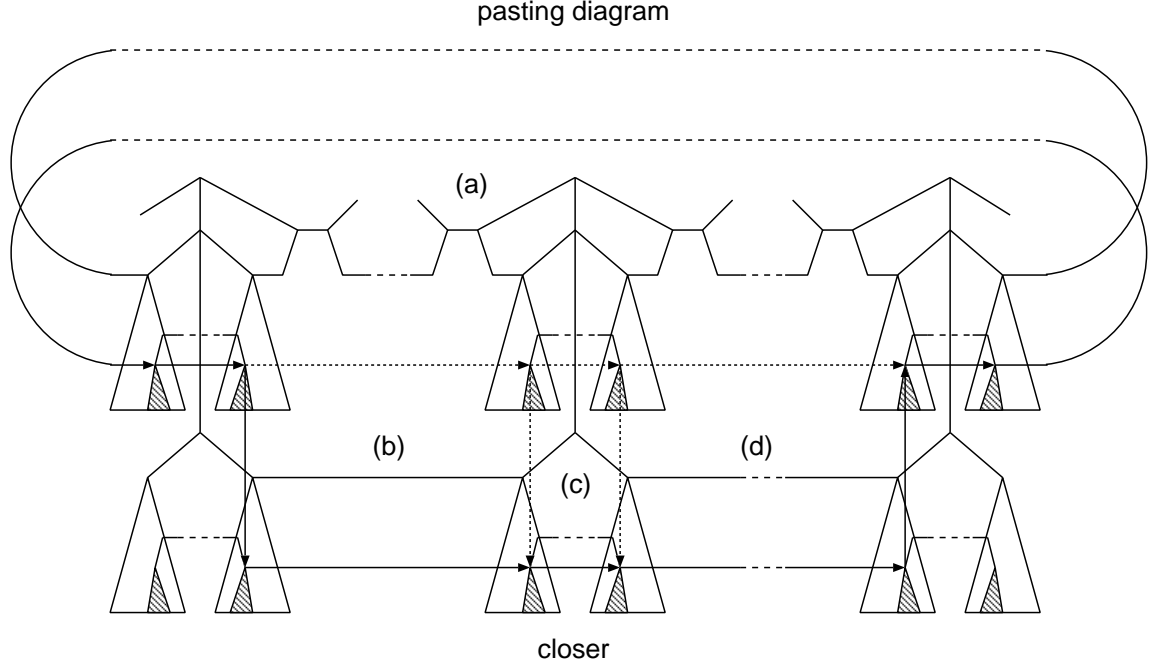


Figure 5: A chain of shell isomorphisms in a closure

Next, we will check the commutativity of links condition for the closure. The following three cases are possible:

1. All links are in  $\Upsilon^\xi$ ;
2. All links are in  $\Upsilon^{\hat{\xi}}$ ;
3. Links in  $\{\langle s_l, t_l \rangle, \langle t_l, s_l \rangle \mid 0 \leq l \leq k\}$  occur.

Commutativity for the case 1 is trivial from the definition of pasting diagrams. For the cases 2 and 3, we show commutativity of a path in Figure 5. The oval path (a) is commutative from the definition of pasting diagrams; the square (b) and (c) is from the construction of closers; and the square (d) is by pasting an alternation of the type (b) and (c) squares. Thus the outer path of arrows in this case is commutative by pasting (a)–(d). In the general case 3, the side trip (b)–(d) might occur several times. The largest roundabouts are paths running only through the closer. This implies commutativity for the case 2, that is, the commutativity of links condition for the closer.  $\square$

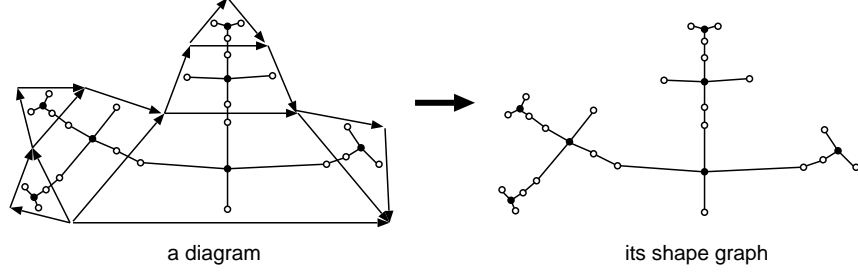


Figure 6: Extraction of shape graphs

## 6 Directed $\omega$ -hypergraphs

### 6.1 directed $\omega$ -hypergraphs

**Definition 6.1 (shape graph)** The (undirected) *shape graph*  $\langle N_0^\xi, N_1^\xi E_0^\xi, E_1^\xi \rangle$  of an  $n$ -frame  $\xi$  is defined as follows:

- the body node set  $N_0^\xi$  is  $S_n^\xi$ ;
- the foot node set  $N_1^\xi$  is  $S_{n-1}^\xi$ ;
- the leg edge set  $E_0^\xi$  is  $\{\{s, t\} \mid s \in S_n^\xi, t \in S_{n-1}^\xi, \pi^\xi(t) = s\}$ ;
- the link edge set  $E_1^\xi$  is  $\{\{t, t'\} \mid \langle t, t' \rangle \in \Upsilon_{n-1}^\xi\}$  (note that  $\pi^\xi(t) \neq \pi^\xi(t')$  from the definition of  $\Upsilon_{n-1}^\xi$ ).

If  $s \in N_0^\xi$ ,  $t \in N_1^\xi$  and  $\{s, t\} \in E_0^\xi$  then we call  $t$  a *foot* of  $s$  and  $\{s, t\}$  a *leg* of  $s$ . The *shape graph*  $\langle N_0^\zeta, N_1^\zeta E_0^\zeta, E_1^\zeta \rangle$  of an  $n$ -frame  $\zeta$  is the shape graph of  $\zeta$ .

The *shape graph* of  $n$ -cell shells or  $n$ -cell diagrams is defined as a special case of  $n$ -frame shells or  $n$ -frames.

**Definition 6.2 ( $n$ -directed  $n$ -frame)** An  $n$ -frame  $\zeta$  is  *$n$ -directed* if it satisfies the following conditions:

- *headedness*: for exactly one  $s \in S_n^\zeta$  and every other  $s' \in S_n^\zeta$ , either  $\lambda_n^\zeta(s)$  is positive and  $\lambda_n^\zeta(s')$  is negative or  $\lambda_n^\zeta(s)$  is negative and  $\lambda_n^\zeta(s')$  is positive, where  $s$  is called the *positive* or *negative head* of  $\zeta$ , respectively;
- *connectedness*: its shape graph is connected;
- *acyclicity*: the graph obtained from its shape graph by getting rid of a body node corresponding to the head, its legs and feet, and link edges connected to them, is acyclic (indeed, this graph is a tree).

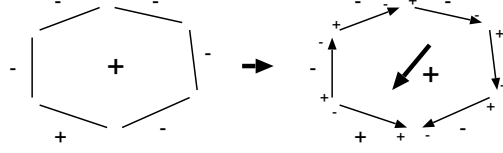


Figure 7: Direction via polarity

An  $n$ -frame with the positive head is said to be *positively  $n$ -directed*, and that with the negative head be *negatively  $n$ -directed*. The same  $n$ -frame can be both positively and negatively  $n$ -directed. An  $n + 1$ -cell whose boundary is such an  $n$ -frame is called a *simple  $n + 1$ -cell*.

**Definition 6.3 (directed  $n$ - and  $\omega$ -hypergraph)** A *directed  $\omega$ -hypergraph* is an  $\omega$ -hypergraph which satisfies the following condition:

- *directedness*: for each  $i \geq 1$ , the boundary of any positive  $i$ -cell is a positively  $i - 1$ -directed  $i - 1$ -frame, and that of any negative  $i$ -cell is a negatively  $i - 1$ -directed  $i - 1$ -frame.

For each  $n \geq 0$ , a *directed  $n$ -hypergraph* is also defined as an  $n$ -hypergraph satisfying the same condition.

## 6.2 directed shells

In the category theory based on  $n$ - or  $\omega$ -hypergraphs, *directed  $n$ -cell shells* and *directed  $n$ -frame shells* play the role of shape diagrams in the usual theory. They are defined by adding some conditions to the induction step of the definitions of  $n$ -cell shells and  $n$ -frame shells

**Definition 6.4 (directed  $n$ -cell shell)** An additional condition is as follows:

- *directedness*: If  $\epsilon(r) = 1$ , then  $\theta|_r$  is a positively directed  $n - 1$ -frame shell and if  $\epsilon(r) = -1$ , then it is a negatively directed one.

**Definition 6.5 (directed  $n$ -frame shell)** Additional conditions are as follows:

- *headedness*: for exactly one  $s \in S_n^\xi$  and every other  $s' \in S_n^\xi$ , either  $\epsilon_n^\xi(s) = 1$  and  $\epsilon_n^\xi(s') = -1$  or  $\epsilon_n^\xi(s) = -1$  and  $\epsilon_n^\xi(s') = 1$ , where  $s$  is called the *positive* or *negative head* of  $\xi$ , respectively;
- *connectedness*: its shape graph is connected;
- *acyclicity*: the graph obtained from its shape graph by getting rid of a body node corresponding to the head, its legs and feet, and link edges connected to them, is acyclic (indeed, this graph is a tree).



An  $n$ -frame shell with the positive head is said to be *positively directed*, and that with the negative head be *negatively directed*. The same  $n$ -frame shell can be both positively and negatively directed. If for an  $n + 1$ -cell shell  $\theta$  with root  $r$ ,  $\theta|_r$  is such an  $n$ -frame shell, then it is called a *simple  $n + 1$ -cell shell*.

**Proposition 6.1** *An  $n$ - or  $\omega$ -hypergraph  $\langle \Sigma, \partial \rangle$  is a directed  $n$ - or  $\omega$ -hypergraph iff for each positive  $i$ -cell  $c$ ,  $\partial_i(c)$  is a positively directed  $i - 1$ -frame shell and for each negative  $i$ -cell, it is a negatively directed one.*

**Proof** By induction on dimensions. □

**Definition 6.6 (directed  $n$ -frame and directed  $n$ -cell diagram)** An  $n$ -frame  $\zeta$  is a *positively* or *negatively directed  $n$ -frame* if  $\zeta$  is a positively or negatively directed  $n$ -frame shell, respectively. Also a *positively* or *negatively directed  $n$ -cell diagram* is defined in the same way.

**Corollary 6.2** *In any directed  $n$ - or  $\omega$ -hypergraph, an  $n$ -frame is a positively or negatively directed  $n$ -frame iff it is an positively or negatively  $n$ -directed  $n$ -frame, respectively.*

**Definition 6.7 ( $\mathbf{DFrm}_k$ )** For a directed  $n$ - or  $\omega$ -hypergraph, the category (groupoid) whose objects are all directed  $k$ -frames and whose arrows are all isomorphisms is denoted by  $\mathbf{DFrm}_k$ . The collection of all directed  $k$ -frames is also denoted by  $\mathbf{DFrm}_k$ .

In the rest of this paper, we will mainly use the usual diagrammatic notations for shells and diagrams (Figure 7).

### 6.3 examples

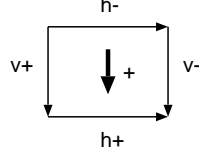
**Example 6.1 (hypergraph in rewriting)** An (directed) *hypergraph* used in hypergraph rewriting [13] is a directed 1-hypergraph.

**Example 6.2 ( $\omega$ -multigraph)** An  $\omega$ -*multigraph* is a directed  $\omega$ -hypergraph  $\langle \Sigma, \partial \rangle$  such that  $\Sigma_{0,1}$  is a singleton set and that any  $c \in \Sigma_{1,1}$  is a simple 1-cell.

**Example 6.3 (doublegraph)** Doublegraphs are underlying graph-like structures for double categories. They are obtained by splitting 1-cells into vertical cells and horizontal cells:

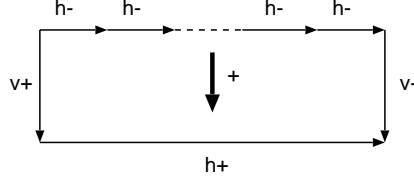
$$\begin{aligned}\Sigma_0 &= \Sigma_{0,1} \amalg \Sigma_{0,-1} \\ \Sigma_1 &= \Sigma_{1,1} \amalg \Sigma_{1,-1} \\ \Sigma_{1,1} &= \Sigma_{1,1}^v \amalg \Sigma_{1,1}^h \\ \Sigma_{1,-1} &= \Sigma_{1,-1}^v \amalg \Sigma_{1,-1}^h\end{aligned}$$

and if  $c \in \Sigma_{1,k}$ , then  $c^* \in \Sigma_{1,-k}$ , etc. 2-cells are as follows:



This approach can be easily extended to multiple categories.

**Example 6.4 (fc-multigraph)** **fc**-multigraphs are underlying graph-like structures for **fc**-multicategories introduced by T. Leinster [8]. Similar notions also appear in [7]. They are a mixture of 2-multigraphs and double graphs. 0-cells and 1-cells are the same as double graphs. 2-cells are as follows



## 6.4 directed pasting shells and diagrams

**Definition 6.8 (boundary graphs)** The *shape graph* of an  $n$ -pasting shell  $\xi$  is defined in the same way of  $n$ -frame shells. The *boundary graph* of an  $n$ -pasting shell  $\xi$  is defined as the shape graph of  $(\hat{\xi}|_{r,\xi})^*$ .

Note that the *boundary graph* of an  $n$ -cell shell  $\theta$  as a special case of  $n$ -pasting shells, matches with the shape graph of  $\theta|_{r,\theta}$ .

**Definition 6.9 (directed  $n$ -pasting shells)** A *directed  $n$ -pasting shell*  $\xi$  is an  $n$ -pasting shell consisting of directed  $n$ -cell shells satisfying the following conditions:

- *homogeneity*: it is positive or negative as an  $n$ -pasting shell.
- *connectedness*: its shape graph is connected;
- *acyclicity*: its shape graph is acyclic (indeed, this graph is a tree).

**Definition 6.10 (directed  $n$ -pasting diagrams)** A *directed  $n$ -pasting diagram* is an  $n$ -pasting diagram whose base  $n$ -pasting shell is a directed  $n$ -pasting shell. A *positive* or *negative* directed  $n$ -pasting diagram is trivially defined, respectively.

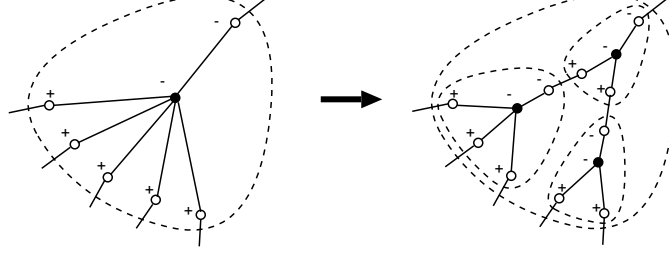


Figure 8: replacement of the boundary graph

**Proposition 6.3** *For any directed  $n$ -frame  $\zeta$  and its head node  $h$ , we can uniquely split it into an directed  $n$ -cell diagram  $\text{cod}(\zeta)$ , called the codomain of  $\zeta$ , and an  $n$ -pasting diagram  $\text{dom}(\zeta)$ , called the domain of  $\zeta$ , where*

- $\text{cod}(\zeta) = \zeta|_h$
- $\text{dom}(\zeta)$  is obtained by deleting, from the data of  $\zeta$ ,  $\zeta|_h$  and indice  $\langle s, s' \rangle$  of which  $s$  or  $s'$  is in  $\Upsilon^{\zeta|_h}$ .

*Of course, if  $\zeta$  is positively directed, then  $\text{cod}(\zeta)$  is positive and  $\text{dom}(\zeta)$  is negative, and if negatively directed, then  $\text{cod}(\zeta)$  is negative and  $\text{dom}(\zeta)$  is positive.*

**Proposition 6.4 (closure of a directed  $n$ -pasting shell)** *For any negative (resp. positive) directed  $n$ -pasting shell  $\xi$ , (1) its positive (resp. negative) closer  $\hat{\xi}$  is a directed  $n$ -cell shell and (2) its positive (resp. negative) closure  $\bar{\xi}$  is a directed  $n$ -frame shell.*

**Proof** (1) We have to check that  $\hat{\xi}|_{r,\xi}$  is a positively directed  $n-1$ -frame shell. (i) *Headedness*: From homogeneity, connectedness and acyclicity of  $\xi$ , the set  $O_\xi$  of open nodes of  $\xi$  contains just one positive node. Therefore  $\hat{\xi}|_{r,\xi}$  contains just one negative node. (ii) *Connectedness and acyclicity*: By induction on the construction of directed  $n$ -pasting shells. (a) The boundary graph of a directed  $n$ -cell shell satisfies the connectedness and acyclicity conditions. (b) If you make a directed  $n$ -pasting shell  $\xi'$  by linking a directed  $n$ -pasting shell  $\xi$  and a directed  $n$ -cell shell  $\theta$  at one open  $n-1$ -node (with satisfying the homogeneity, connectedness and acyclicity for  $\xi'$ ), then replacement of the boundary graph occurs (Figure 8) and the resulting boundary graph of  $\xi'$  also satisfies the connectedness and acyclicity. The acyclicity is obtained by reduction to absurdity. Suppose the existence of cycles and consider the graph obtained by deleting link edges of cycles from the boundary graph, and its polarity of foot nodes, it contradicts the directedness of the boundary  $n-1$ -frame shell  $\theta|_{r,\theta}$  of each  $n$ -cell shell  $\theta$  in the directed  $n$ -pasting shell. (c) Any directed  $n$ -pasting shell can be

constructed by finitely iterating this process, and then the boundary graph of the closer satisfies the connectedness and acyclicity. Thus  $\hat{\xi}|_{r\hat{\xi}}$  is a positively directed  $n - 1$ -frame shell. Since  $\epsilon^{\hat{\xi}}(r^{\hat{\xi}}) = 1$ ,  $\hat{\xi}$  is a positively directed  $n$ -cell shell. The negative case is in parallel.

(2) From the homogeneity, connectedness and acyclicity of  $\xi$ , the headedness, connectedness and acyclicity of  $\bar{\xi}$  is obvious.  $\square$

## 7 Weak $\omega$ -categories

### 7.1 $\omega$ -identity, $\omega$ -invertibility and $\omega$ -universality

We will *coinductively* define three notions:  $\omega$ -identity,  $\omega$ -invertibility and  $\omega$ -universality. All  $n$ -dimensional notions depend on  $n + 1$ - or  $n + 2$ -dimensional ones. The reader unfamiliar with coinductive definitions may think of only the case in which coinduction steps terminate.

One source of our idea is Michael Makkai's work on anabategories [9]. At a glance, as Makkai pointed out in [10], *saturated* anabategories could be regarded as 2-dimensional weak categories of Baez-Dolan. But we don't think of them to be equivalent notions for some reasons:

1) In anabategories, two composite arrows for the same composable sequence of arrows are equivalent. While in Baez-Dolan's there are two opposite universal 2-cells between two composite arrows by virtue of balancedness, no explicit relation between them appears. In fact, we can prove that they are equivalences in a sense, because Baez and Dolan only think of finite dimensional cases. But we cannot prove it in that way for infinite dimensional cases. Therefore we need to characterize those opposites as a sort of equivalences.

2) Different from anabategories, composites of empty sequence are introduced in Baez-Dolan's and expected to play the role of identities. But as well as the above, we cannot prove the property of identity in infinite dimensional cases. Thus we also have to define identities explicitly.

3) In (not necessarily saturated) anabategories, an object isomorphic to a composite might not be a composite. It suggests that if we introduce a sort of equivalences, it is natural to treat equivalences and composition separately and add a saturatedness condition.

**Definition 7.1 ( $\omega$ -identical cells)** An  $n$ -cell  $c$  is  $\omega$ -identical if it is simple and satisfies  $\text{dom}(c) \cong \text{cod}(c)$  and the following conditions:

$$\begin{array}{ccc} \forall f & & \exists \alpha \\ \begin{array}{c} \nearrow c \\ \searrow f \\ \xrightarrow{f} \end{array} & \Rightarrow & \begin{array}{c} \nearrow c \\ \searrow f \\ \xrightarrow{f} \\ \Downarrow \alpha \end{array} \end{array} \quad \alpha: \omega\text{-universal}$$

and

$$\begin{array}{ccc} \forall g & & \exists \beta \\ \begin{array}{c} \begin{array}{ccc} & \nearrow g & \searrow c \\ & & \downarrow \beta \\ & \xrightarrow{g} & \end{array} \end{array} & \Rightarrow & \begin{array}{c} \begin{array}{ccc} & \nearrow g & \searrow c \\ & & \downarrow \beta \\ & \xrightarrow{g} & \end{array} \end{array} \end{array} \quad \beta: \omega\text{-universal}.$$

**Definition 7.2** ( $\omega$ -invertibility,  $\omega$ -equivalence,  $\simeq$ ) A pair of  $n$ -cells  $f$  and  $g$  is an  $\omega$ -invertible pair if both  $f$  and  $g$  are simple and satisfy  $\text{dom}(f) \cong \text{cod}(g)$  and  $\text{dom}(g) \cong \text{cod}(f)$  and the following conditions:

$$\begin{array}{ccc} \exists \alpha, i & & \\ \begin{array}{c} \begin{array}{ccc} & \nearrow g & \searrow f \\ & & \downarrow \alpha \\ & \xrightarrow{i} & \end{array} \end{array} & i: \omega\text{-identical}, & \end{array}$$

and

$$\begin{array}{ccc} \exists \beta, j & & \\ \begin{array}{c} \begin{array}{ccc} & \nearrow f & \searrow g \\ & & \downarrow \beta \\ & \xrightarrow{j} & \end{array} \end{array} & j: \omega\text{-identical}. & \end{array}$$

$f$  and  $g$  are called  $\omega$ -invertible. We say that two  $n-1$  cells  $\lambda(r^{\text{dom}(f)})$  and  $\lambda(r^{\text{cod}(f)})$  are  $\omega$ -equivalent and write it as  $\lambda(r^{\text{dom}(f)}) \simeq \lambda(r^{\text{cod}(f)})$ .

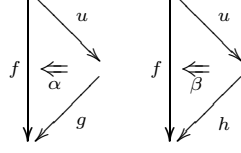
**Definition 7.3** ( $\omega$ -universal cells) An  $n$ -cell  $u$  is  $\omega$ -universal if for any  $n$ -cell  $f$  of

$$\begin{array}{c} \searrow u \\ \downarrow f \end{array}$$

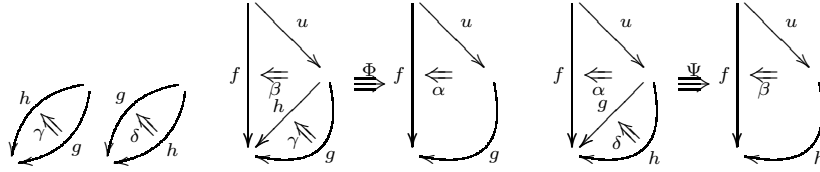
there exist an  $n$ -cell  $g$  and an  $\omega$ -universal  $n+1$ -cell  $\alpha$  to be

$$\begin{array}{c} \searrow u \\ \begin{array}{c} \Leftarrow \alpha \\ \nearrow g \end{array} \\ \downarrow f \end{array}$$

and for two such pairs,  $g$  and  $\alpha$ ,  $h$  and  $\beta$



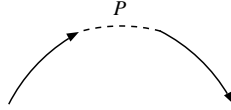
there exist an  $\omega$ -invertible pair of  $n + 1$ -cells,  $\gamma$  and  $\delta$ , and two  $\omega$ -universal  $n + 2$ -cells,  $\Phi$  and  $\Psi$ , such that



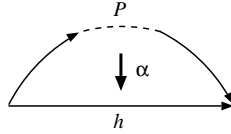
## 7.2 weak $\omega$ -categories

**Definition 7.4 (weak  $\omega$ -categories)** A directed  $\omega$ -hypergraph is a *weak  $\omega$ -category* if it satisfies the following conditions:

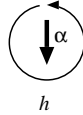
- *existence of closers and occupants:* For any  $n$ -pasting diagram  $P$ ,



there exist an  $\omega$ -universal  $n + 1$ -cell  $\alpha$  and an  $n + 1$ -cell  $h$  such that  $\text{dom}(\alpha) \cong_n P$  and  $\lambda(r^{\text{cod}(\alpha)}) = h$ :

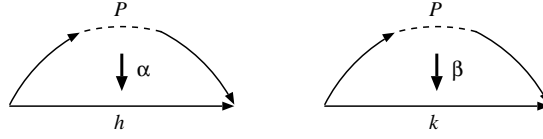


Moreover, if  $P$  is an empty pasting diagram, then for each  $n - 1$ -cell diagram  $x$ , there exist such  $\alpha$  and  $h$  satisfying  $\text{dom}(h) \cong_{n-1} \text{cod}(h) \cong_{n-1} x$ .

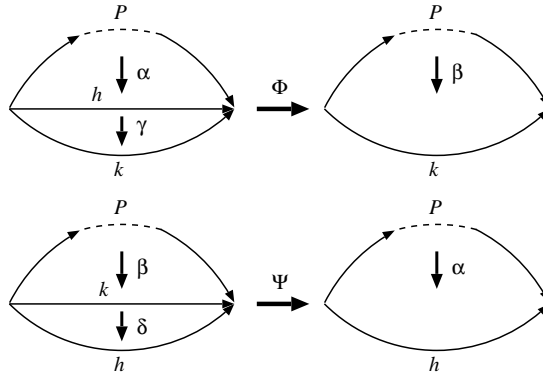


We call  $\alpha$  an *occupant* for  $P$  and  $h$  a *closer* of  $P$ .

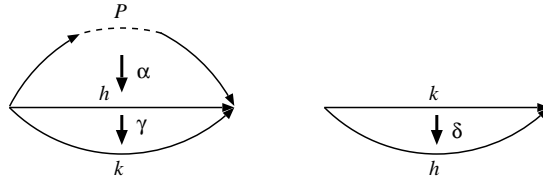
- *weak uniqueness of closers and occupants*: For two such pairs as above,  $\alpha$  and  $h$ ,  $\beta$  and  $k$



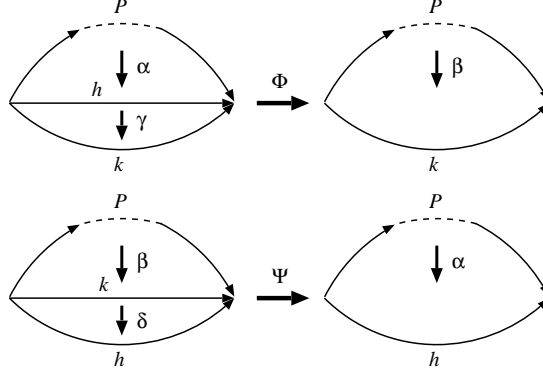
there exist an  $\omega$ -invertible pair  $\gamma$  and  $\delta$  and two  $\omega$ -universal cells  $\Phi$  and  $\Psi$  such that



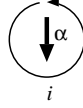
- *saturation of closers and occupants*: For an  $\omega$ -universal cell  $\alpha$  and an  $\omega$ -invertible pair  $\gamma$  and  $\delta$



there exists  $\omega$ -universal cells  $\beta, \Phi, \Psi$  such that



- *$\omega$ -identical closers (1)*: For any  $\omega$ -universal  $n + 1$ -cell  $\alpha$  as follows, an  $n$ -cell  $i$  is  $\omega$ -identical:



- *$\omega$ -identical closers (2)*: For any  $\omega$ -identical  $n$ -cell  $i$ , there is an  $\omega$ -universal  $n + 1$ -cell  $\alpha$  as above.
- *$\omega$ -universal closers*: Any closer for a  $n$ -pasting diagram made of  $\omega$ -universal  $n$ -cells is  $\omega$ -universal.

A weak  $\omega$ -categories is defined when this coinductive definition makes sense.

**Proposition 7.1** *An  $\omega$ -identical cell is  $\omega$ -invertible.*

**Proposition 7.2** *Every two  $\omega$ -identical cells are  $\omega$ -equivalent.*

Again, the reader unfamiliar with coinduction may think of weak  $n$ -categories.

**Definition 7.5 (weak  $n$ -categories)** A weak  $\omega$ -category is a *weak  $n$ -category* if for each  $k$  higher than  $n$ , all simple  $k$ -cells are  $\omega$ -invertible.

From the axioms above, we can recognize that identity is independent from the definition of composition. In fact, to define our weak  $\omega$ -categories, we can exclude empty pasting diagram and related axioms and add an axiom for existence of  $\omega$ -identical  $n + 1$ -cells for each  $n$ -cells. This is slightly simpler than those defined above. And we conjecture that our definition would be equivalent to that of J. Penon [12], and furthermore that if we abandon saturatedness and for each  $n$ -cell we choose just one  $\omega$ -identical  $n + 1$ -cell whose domain and



codomain are that  $n$ -cell, then the category of our small weak  $\omega$ -categories (with suitable functors) would be isomorphic to the category of Penon's.

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